Diffusion process with two reflecting barriers in a time-dependent potential

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We consider a Brownian particle which is driven by a harmonically oscillating force, the motion of which is restricted by two reflecting boundaries. We solve the Fokker-Planck equation using the finite-element method and focus on the dynamics of the mean position of the particle in the time-asymptotic regime. As a function of the strength of the external force, the response of the system, i.e., the amplitude of the mean position and the dynamical shift, in the stationary limit shows a resonancelike behavior as a function of the diffusion coefficient for certain parameter regimes. We explain these numerical results heuristically and give some qualitative estimates.

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I. INTRODUCTION

Noise plays an important role in the behavior of dynamical systems. In recent years noise-induced phenomena, such as noise-induced transitions, stochastic resonance, or the physics of Brownian motors, are of increasing interest in physics, chemistry, and biology [1–4]. In the context of stochastic resonance the dynamics of diffusion processes in time-dependent potentials has been the subject of recent studies [5]. Only a few systems are exactly solvable [6–9]. In most cases approximate procedures like two-state approximations, assumptions of large or small frequencies of the external signal, or linear response theory (i.e., a perturbation expansion in terms of the amplitude of the external driving field) are necessary [10–18].

The studies presented in this paper have been stimulated by experimental investigations into various classes of soft matter systems such as colloidal dispersions, polymer melts or solutions, membranes, and biological macromolecules. The typical length scales of these systems are in the mesoscopic regime between atomic sizes and macroscopic scales. Colloidal systems are ideal model systems for problems in statistical physics. The mesoscopic particles are constantly bombarded by the random impacts of the molecules of the liquid. Therefore they undergo a Brownian motion. With sizes comparable to the wavelength of the visible light, such systems can be investigated by convenient optical methods and trajectories of individual colloidal particles can be observed directly. The behavior of colloidal dispersions in different confining geometries and under the influence of different external fields are of growing interest [19,20]. It is this context of research activities in which we see our paper. Blickle *et al.* studied the thermodynamics of single colloidal trajectories in a time-dependent nonharmonic potential [21]. They used an aqueous suspension of highly charged polystyrene beads, which were illuminated with light. The particle concentration was sufficiently low to guarantee that there was only a single particle within the field of view. Among others they measured the distance probability distribution of a colloid in front of a wall to get particle-wall potentials.

We are now interested in the one-dimensional diffusive motion of a charged particle in a linear time-dependent potential restricted by two reflecting boundaries. The corresponding external electric force consists of two parts, a time-independent part pushing the particle to the right boundary and an oscillating part. Differently charged capacitor plates can provide this time-independent electric force. In addition an oscillating electric field is switched on. With respect to the experiments referred to above, our theoretical studies describe a slightly modified situation: single charged particles move not towards a single wall but between the two plates of a capacitor. As seen in the following under certain conditions the response of the system shows resonancelike behavior, which corresponds to a maximum of energy trapped in the system. We solve the Fokker-Planck equation (FPE) with the finite-element method (FEM) [22–25]. From the numerical solution of the FPE we obtain the distribution function and calculate the mean position of the particle for different diffusion coefficients and different amplitudes of the oscillating force, called force strengths in the following. It is shown that the mean position is an oscillating function. Depending on the diffusion coefficient and the strength of the external oscillating force these oscillations are more or less symmetric with respect to the time-averaged mean position. In the time-

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asymptotic limit we consider the variation of the stationary amplitude of the mean position with the diffusion coefficient for different strengths of the external time-dependent force. We discuss the resonancelike behavior of the system response as a function of the diffusion coefficient. The occurrence of the resonance and also its shape are controlled by the strength of the external oscillating force. Our numerical results show that there is a certain force stength, called critical force strength, above which the resonance behavior is lost.

It seems impossible to derive an exact analytical solution of the time-dependent mean position as a function of the diffusion coefficient in the general case. Only in the deterministic case, i.e., in the limit of vanishing diffusion coefficient, the time-dependent position of the particle can be calculated exactly. Furthermore, for small values of the diffusion coefficient we give qualitative estimates of the variation of the amplitude of the mean position with the diffusion coefficient. In this way, we can predict the existence of the resonance, but we cannot determine its position (in terms of the diffusion coefficient) analytically. We show that for the critical force the nondiffusive particle impinges on the left boundary for the first time when the force strength is increased. Finally we consider the dynamical shift of the mean position, which is defined as the difference between the averaged and the equilibrium mean position. The dynamical shift is always negative. The time-averaged mean position is smaller than the equilibrium mean position for the unperturbed potential and the particle is thus shifted against the unperturbed potential to higher energies. In this way work of the time-dependent force is stored in the system. The dynamical shift shows a similar resonance behavior as the amplitude of the mean position. For a certain parameter range of the oscillating force strength there is a diffusion coefficient for which a maximum of energy is trapped in the system. Qualitative estimates show that the critical force strength for the dynamical shift is the same as for the amplitude of the mean position.

Both the method used and the model studied in the present paper are basically different from those in our previous publications, [26,27]. In these papers, we have analyzed diffusion restricted onto a semi-infinite domain. In [26], we have formulated and solved numerically a Volterra integral equation of the first kind which contains the final space-resolved dynamics of the concentration profile. In [27], we have developed a special time-asymptotic procedure yielding the harmonic decomposition of the long-time nonlinear response. However, neither method is suitable for the present two-boundary problem and, moreover, the second reflecting boundary introduces qualitatively new features of the resulting dynamics. Namely, the above-described critical force does not occur in the one-boundary problem.

The paper is structured as follows: We start with the theoretical description of our model in Sec. II and introduce the quantities of interest. In Sec. III we show our numerical results. The mean position and the time-averaged mean position in dependence on the diffusion coefficient and for various strengths of the external oscillating force as well as the amplitude of the mean position and the dynamical shift are presented. In the following Sec. IV the resonancelike behavior of both the amplitude of the mean position and the dynamical shift are discussed in detail. In the concluding section, Sec. V, we summarize our results.

II. MODEL

We consider a particle which diffuses in a one-dimensional domain $x \in [x_1, x_2]$ restricted by two reflecting boundaries at $x = x_1$ and $x = x_2$, respectively. The time evolution of the probability density for the particle position is controlled by the one-dimensional Fokker-Planck (Smoluchowski) equation

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x} j(x,t). \tag{1}$$

Here $p(x,t)$ is the probability density to find the particle at the point x at time t and

$$j(x,t) = -D\frac{\partial}{\partial x} p(x,t) - \frac{1}{\Gamma}\left[\frac{\partial V(x,t)}{\partial x}\right] p(x,t) \tag{2}$$

is the corresponding probability current. Γ is the particle mass times the friction coefficient. $D = k_B T/\Gamma$ is the diffusion coefficient which describes the strength of the thermal noise. $V(x,t)$ denotes the time-dependent potential, i.e., $F(x,t) = -\partial V(x,t)/\partial x$ is the corresponding force. Because of the two reflecting boundaries, the probability density $p(x,t)$ has to fulfill the normalization condition $\int_{x_1}^{x_2} p(x,t)dx = 1$. At the same time, the probability current $j(x,t)$ has to vanish at the boundaries. Differently speaking, we require the boundary conditions

$$\left.\frac{\partial}{\partial x} p(x,t)\right|_{x=x_i} = -\frac{1}{\Gamma D}\left.\frac{\partial V(x,t)}{\partial x} p(x,t)\right|_{x=x_i} \tag{3}$$

for $i = 1, 2$, and for any $t \geq 0$. The dynamical equation (1) must be supplemented by the initial condition. In our numerical investigations we choose it in the specific form,

$$p(x,0) = \frac{1}{l}\left\{1 - \cos\left[\frac{2\pi(x - x_1)}{l}\right]\right\} \tag{4}$$

for $x \in [x_1, x_2]$, and $p(x,0) = 0$ for $x \notin [x_1, x_2]$. Here $l = x_2 - x_1$ denotes the width of the diffusion domain. The function $p(x,0)$ is properly normalized and it is consistent with the required boundary conditions. Furthermore, we assume that the diffusing particle is influenced by a spatially homogeneous and harmonically oscillating force. Therefore the potential in Eq. (2) has the following form:

$$V(x,t) = -xF(t) = -x[F_0 - F_1 \sin(\omega t)]. \tag{5}$$

We always assume $F_0 \geq 0$ and $F_1 \geq 0$. If the ratio F_1/F_0 exceeds unity, the total force can change its sign, pushing the particle to the left or to the right.

An important quantity which characterizes the emerging dynamics is the mean position of the particle $M(t)$. It is defined as a stochastic average, i.e., by the formula

$$M(t) = \int_{x_1}^{x_2} x p(x,t) dx. \tag{6}$$

Our numerical results show that after some transitory time, the mean position exhibits periodic oscillations with the period of the external force. We are primarily interested in the time-asymptotic regime and we now introduce two quantities which characterize the stationary regime. First, we determine the maximum ($M_{\max}$) and the minimum ($M_{\min}$) of the mean position within one period of the oscillations in the stationary regime and calculate the quantity

$$A_{\text{stat}} = \frac{1}{2}(M_{\max} - M_{\min}). \tag{7}$$

It measures the amplitude of the mean position oscillations. Second, we introduce the time-averaged mean position

$$M_{\text{av}} = \lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} M(t')dt', \tag{8}$$

where $T=2\pi/\omega$ denotes the period of the driving force. It turns out that this quantity is shifted relative to the equilibrium mean position M_{eq} of the particle under the only influence of the time-independent force F_0. This equilibrium mean position is given by the expression

$$M_{\text{eq}} = \lim_{t\to\infty} M(t) = x_1 + l\left[\frac{1}{1-\exp(-la)} - \frac{1}{la}\right], \tag{9}$$

which is obtained by solving Eq. (2) for $j(x,t)=0$ and calculating the average (6), where $a=F_0/(\Gamma D)$. In order to analyze the shift in question, we consider the difference between the time-averaged mean position (8) in the nonequilibrium stationary regime and the equilibrium position (9) in the static case. We call the quantity

$$S = M_{\text{av}} - M_{\text{eq}} \tag{10}$$

the dynamical shift. In general the two quantities A_{stat} and S depend on the whole set of model parameters, i.e., on the force parameters F_0, F_1, ω, on the width of the diffusion domain l, and on the diffusion coefficient D. In the next section, we introduce an appropriate set of reduced parameters and discuss the dependencies of A_{stat} and S on them.

III. NUMERICAL RESULTS

To solve the dynamical equation (1) within the space interval $x\in[x_1,x_2]$ and the time interval $t\in[0,t_{\text{end}}]$, we have used the finite-element method (FEM) [25]. We shortly outline just the main steps of the numerical procedure.

We concentrate on the solution of Eq. (1) within the space interval $x\in[x_1,x_2]$, and in the time interval $t\in[0,t_{\text{end}}]$. First, we multiply the FPE (1) by an arbitrary test function $w(x)$ which vanishes at the boundaries. Then we integrate both sides of the emerging equation over the whole space interval $[x_1,x_2]$. As a result, we get the so-called weak form of the differential equation. This weak form is the basis for the FEM. Afterwards, we discretize the space in N_x finite elements with N_x+1 nodes $\{i\}$. For each space element we make a linear ansatz for the test function $w(x)$ and the solution $p(x,t)$. After some manipulations we get a system of differential equations, the solution of which yields the probability density $\{p_i(t)\}$ at each node i. Up to now the time was continuous. Now the time interval $t\in[0,t_{\text{end}}]$ is also divided in N_t equal time intervals with N_t+1 nodes $\{k\}$ and the time derivative of $\{p_i(t)\}$ is replaced by a difference quotient based on the Crank-Nicolson prescription [25]. We arrive at a system of algebraic equations for the values $\{p_i(t_k)\}$ of the probability density at each node $i=1,2,\ldots,N_x+1$, and for each time step t_k, $k=1,2,\ldots,N_t+1$. This system is solved iteratively.

For the numerical procedure, we have introduced the scaled dimensionless space coordinate $\xi=(\omega\Gamma/F_0)x$, and time $\tau=\omega t$. Performing the corresponding substitutions, the dynamical equation (1) takes the scaled form,

$$\begin{aligned}\frac{\partial}{\partial\tau}u(\xi,\tau) &= \tilde{D}\frac{\partial^2}{\partial\xi^2}u(\xi,\tau) - [1-\lambda\sin(\tau)]\frac{\partial}{\partial\xi}u(\xi,\tau)\\ &= \frac{\partial}{\partial\xi}\left[\tilde{D}\frac{\partial}{\partial\xi}u(\xi,\tau) - [1-\lambda\sin(\tau)]u(\xi,\tau)\right],\end{aligned} \tag{11}$$

with $u(\xi,\tau)=[F_0/(\omega\Gamma)]p(x,t)$. In the course of the transformation, there appear two important dimensionless parameters, $\tilde{D}=D\omega\Gamma^2/F_0^2$ and $\lambda=F_1/F_0$. In the following, we shall mainly discuss how the dynamics of the particle changes with the change of these two parameters. Keeping the parameters F_0, ω, and Γ constant, the change of the parameter $\tilde{D}$ corresponds to the change of the environmental temperature. As for the parameter λ, it measures the amplitude of the oscillating force relative to the time-independent component F_0. For the sake of simplicity, we take always $x_1=0$, and hence $x_2=l$. In the following, the dimensionless width of the diffusion domain is described by the parameter $\theta=(\omega\Gamma/F_0)l$. Moreover, we always take $\theta=1$.

The above procedure naturally leads to the (dimensionless) reduced mean position $\mu(\tau)=(\omega\Gamma/F_0)M(t)$ [cf. Eq. (6)]. For completeness, we introduce also the reduced amplitude $\alpha_{\text{stat}}=(\omega\Gamma/F_0)A_{\text{stat}}$ [cf. Eq. (7)], the reduced time-averaged mean position $\mu_{\text{av}}=(\omega\Gamma/F_0)M_{\text{av}}$ [cf. Eq. (8)], the reduced equilibrium mean position $\mu_{\text{eq}}=(\omega\Gamma/F_0)M_{\text{eq}}$ [cf. Eq. (9)], and the reduced dynamical shift $\sigma=(\omega\Gamma/F_0)S$ [cf. Eq. (10)]. In the following, we always use these reduced parameters and omit the specification "reduced."

As for the parameters which are connected with the numerical procedure itself, we always take $N_\xi=510$, $N_\tau=510$, and $\tau_{\text{end}}=32.0$. We now present in the following Secs. III A–III D the numerical results. The discussion is given in Sec. IV.

A. Mean position of the particle

In this subsection, we investigate the time dependence of the particle mean position $\mu(\tau)=\int_0^1\xi u(\xi,\tau)d\xi$. Figures 1–3 illustrate the mean position $\mu(\tau)$ as a function of the time. We show also the corresponding time-averaged mean posi-

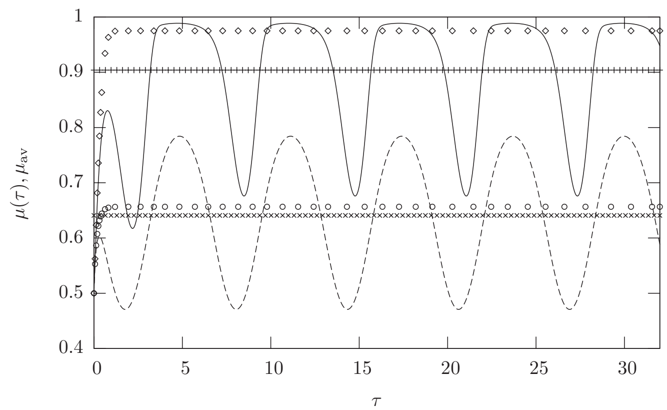

FIG. 1. Mean position $\mu(\tau)$ as a function of time for $\lambda=1.2$: $\tilde{D}=0.025$ (—) and $\tilde{D}=0.5$ (--), respectively. The corresponding time-averaged mean positions $\mu_{\rm av}$ for $\lambda=1.2$: $\tilde{D}=0.025$ (+) and $\tilde{D}=0.5$ (×), respectively, are independent of time [see the definition in Eq. (8)]. For comparison, we plot also the mean position for $\lambda=0.0$: $\tilde{D}=0.025$ (◇) and $\tilde{D}=0.5$ (○), respectively.

tion $\mu_{\rm av}$ and, in Fig. 1, the time evolution of the mean position for the case $\lambda=0$.

First we consider the case $\lambda=0$. The particle, more precisely the average value of the particle position, starts in the middle of the diffusion domain and moves towards the right boundary. After some time an equilibrium distribution is reached giving an equilibrium mean position $\mu_{\rm eq}$. The expression of $\mu_{\rm eq}$ has been obtained by adapting Eq. (9) and using $\theta=(\omega\Gamma/F_0)l=1$

$$\mu_{\rm eq}=\tilde{D}\frac{1}{\exp(1/\tilde{D})-1}\left[\left(\frac{1}{\tilde{D}}-1\right)\exp\left(\frac{1}{\tilde{D}}\right)+1\right]$$
$$=\frac{1}{1-\exp(-1/\tilde{D})}-\tilde{D}. \qquad (12)$$

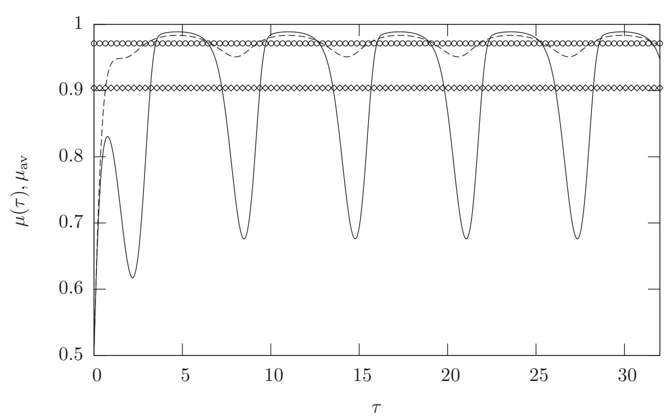

FIG. 2. Mean position $\mu(\tau)$ as a function of time for $\tilde{D}=0.025$: $\lambda=1.2$ (—) and $\lambda=0.5$ (--), respectively. Corresponding time-averaged mean position $\mu_{\rm av}$ for $\tilde{D}=0.025$: $\lambda=1.2$ (◇) and $\lambda=0.5$ (○), respectively.

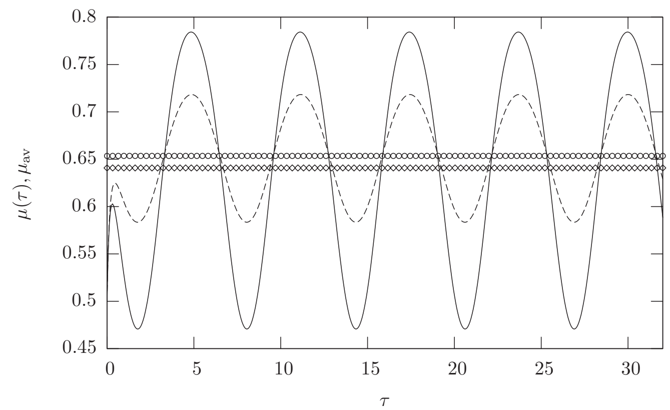

FIG. 3. Mean position $\mu(\tau)$ as a function of time for $\tilde{D}=0.5$: $\lambda=1.2$ (—) and $\lambda=0.5$ (--), respectively. Corresponding time-averaged mean position $\mu_{\rm av}$ for $\tilde{D}=0.5$: $\lambda=1.2$ (◇) and $\lambda=0.5$ (○), respectively.

As follows from Eq. (12), the equilibrium mean position depends on the parameter $\tilde{D}$. Smaller values of the diffusion coefficient $\tilde{D}$ yield larger equilibrium mean positions. Hence the particle remains closer to the right boundary.

Switching on the oscillating component ($\lambda>0$), after a transitory time domain we observe nonlinear oscillations of the mean position with the fundamental period $T=\frac{2\pi}{\omega}$ of the external force. For a low enough diffusion coefficient $\tilde{D}$ (temperature) (cf. the full curve in Fig. 1 with $\tilde{D}=0.025$), the oscillations are strongly asymmetric relative to the time-averaged mean position. Within each period, the particle spends most of the time near the right boundary and makes just short excursions towards the center of the diffusion domain. The oscillations exhibit strong nonlinear features, i.e., the response includes higher harmonics with non-negligible amplitudes. On the other hand, for a large diffusion coefficient (temperature) ($\tilde{D}=0.5$), the oscillations are more symmetric and the time-averaged mean position approaches the center of the diffusion domain. In both cases $\mu_{\rm av}$ is smaller than the mean position for $\lambda=0$ ($\mu_{\rm eq}$).

Having a fixed value of the diffusion coefficient and increasing the parameter λ, the range of the oscillations increases. Compare, e.g., the broken curve in Fig. 2 for $\lambda=0.5$, with the full one for $\lambda=1.2$. The same behavior is demonstrated in Fig. 3. If we increase the parameter λ still further, the right boundary starts to play a decisive role in limiting the motion of the particle. This means that the time-averaged mean position $\mu_{\rm av}$ again approaches the middle of the diffusion domain and the oscillations become symmetric with respect to this value.

B. Time-averaged mean position

Figure 4 shows the time-averaged mean position $\mu_{\rm av}$ as a function of the diffusion coefficient $\tilde{D}$ for different strengths of the oscillating force λ. The time-averaged mean position decreases with increasing $\tilde{D}$ as well as with increasing λ. For $\tilde{D}\to\infty$ it approaches the middle of the diffusion domain. In

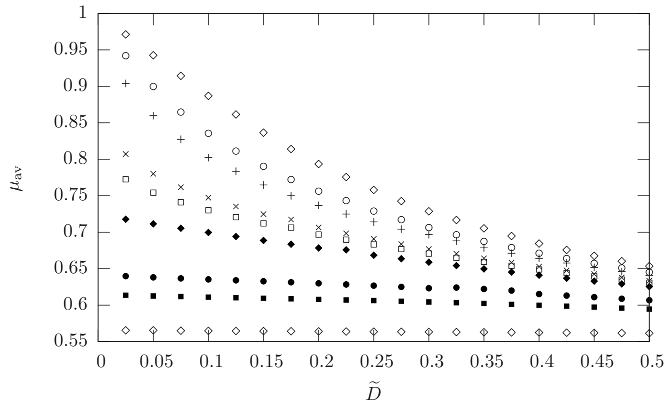

FIG. 4. Time-averaged mean position μ_{av} as a function of the diffusion coefficient $\widetilde{D}$ for several values of λ: λ=5.0 (◇); λ=3.0 (■); λ=2.5 (●); λ=1.8 (◆); λ=1.6 (□); λ=1.5 (×); λ=1.2 (+); λ=1.0 (○); λ=0.5 (◇). (The software repeats symbols after eight steps.)

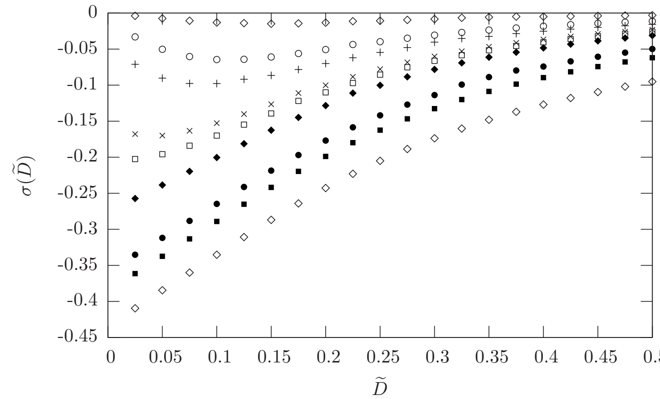

FIG. 6. Dynamical shift $\sigma(\widetilde{D})$ as a function of $\widetilde{D}$ for several values of the parameter λ: λ=5.0 (◇); λ=3.0 (■); λ=2.5 (●); λ=1.8 (◆); λ=1.6 (□); λ=1.5 (×); λ=1.2 (+); λ=1.0 (○); λ=0.5 (◇). (The software repeats symbols after eight steps.)

this limit, the amplitude of the oscillations α_{stat} approaches zero (cf. Fig. 5 in Sec. III C).

C. Amplitude of the mean position

In this section we investigate the dependence of the amplitude of the mean position

$$\alpha_{stat} = \frac{1}{2}(\mu_{max} - \mu_{min}) \tag{13}$$

on the diffusion coefficient $\widetilde{D}$ for different strengths of the oscillating force. There is an upper limit for the value of the amplitude of the mean position given by one-half of the width of the domain ($\alpha_{stat}^{upper}=0.5$). As seen from Fig. 5, the amplitude α_{stat} increases with increasing strength of the external oscillating force. For $\widetilde{D}\to\infty$ the amplitude α_{stat} approaches zero, i.e., the diffusion process determines the motion of the particle and the particle is insensitive to the oscillating external force. It seems that there is a critical force strength $F_1^{crit,\alpha}$ corresponding to a parameter $\lambda^{crit,\alpha}$. For $\lambda<\lambda^{crit,\alpha}$ there is a diffusion coefficient $\widetilde{D}_{max}^{\alpha}$ for which the amplitude α_{stat} takes a maximum. $\widetilde{D}_{max}^{\alpha}$ decreases with increasing λ.

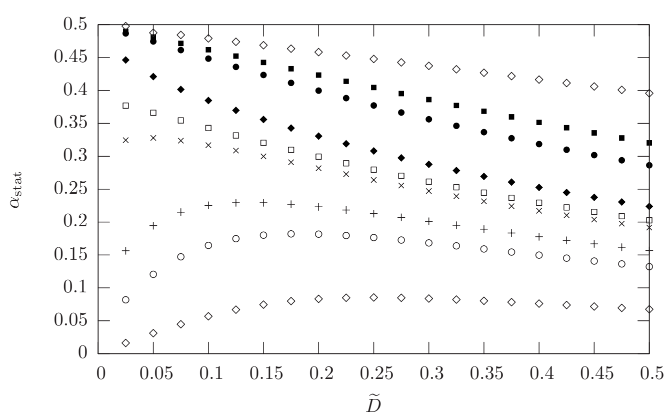

FIG. 5. Amplitude of the mean position α_{stat} as a function of $\widetilde{D}$ for several values of the parameter λ: λ=5.0 (◇); λ=3.0 (■); λ=2.5 (●); λ=1.8 (◆); λ=1.6 (□); λ=1.5 (×); λ=1.2 (+); λ=1.0 (○); λ=0.5 (◇). (The software repeats symbols after eight steps.)

D. Dynamical shift

Figure 6 presents the dependence of the dynamical shift $\sigma=\mu_{av}-\mu_{eq}$ on the diffusion coefficient $\widetilde{D}$ for several parameter values λ. The dynamical shift is always negative and the modulus increases with increasing strength of the oscillating force. In the same way as for the time-averaged mean position, there is also a lower limit for the dynamical shift, $\sigma^{lower}=-0.5$. In the limit $\widetilde{D}\to\infty$ the dynamical shift approaches zero. Both the time-averaged mean position μ_{av} and the equilibrium mean position μ_{eq} take the value 0.5 asymptotically. Analogously to the amplitude of the mean position there are a critical force strength $F_1^{crit,\sigma}$ and a parameter $\lambda^{crit,\sigma}$, respectively. For $\lambda<\lambda^{crit,\sigma}$ there is a diffusion coefficient $\widetilde{D}_{max}^{\sigma}$ for which the dynamical shift takes a minimum.

IV. DISCUSSION

For $\widetilde{D}=0$ the numerical procedure does not allow for a solution of the FPE (11). However, the case $\widetilde{D}=0$ can be investigated by a direct solution of the corresponding deterministic equation. From this solution (together with the solution of the FPE for small values of $\widetilde{D}$) we can qualitatively estimate the behavior of the various variables considered in Sec. III. Therefore we first calculate for $\widetilde{D}=0$ the time-averaged mean position μ_{av} and the amplitude of the mean position α_{stat} from the deterministic equation of motion,

$$\frac{d}{d\tau}\xi(\tau) = 1 - \lambda \sin(\tau). \tag{14}$$

In the case $\widetilde{D}=0$, the dynamical shift is given by the difference of the time-averaged mean position μ_{av} and the equi-

TABLE I. Time-averaged mean position μ_{av} for $\tilde{D}=0$ and for several values of the parameter λ.

λ	1.2	1.5	1.6	1.8	2.5	3.0	5.0
μ_{av}	0.98	0.87	0.82	0.72	0.64	0.61	0.57

librium position $\mu_{eq}=1$. For a more detailed discussion of the dependence of the amplitude of the mean position α_{stat} and of the dynamical shift σ on the diffusion coefficient $\tilde{D}$ we perform some qualitative estimates.

A. Time-averaged mean position

Figure 4 shows the time-averaged mean position μ_{av} as a function of the diffusion coefficient $\tilde{D}$ for various strengths of the oscillating force $\lambda=F_1/F_0$. In the case $\tilde{D}=0$, the time-averaged mean position is given by the exact solution of the deterministic equation of motion (14). Depending on the strength of the oscillating force, we can distinguish three regimes.

Case $\lambda\leqslant 1$. The particle moves in an oscillatory way towards the right boundary. Once it reaches the boundary, it sticks to it. In the time-asymptotic limit, the time-averaged mean position equals 1.

Case $1<\lambda\leqslant\lambda^*$. Consider the following scenario. The particle first moves towards the right boundary as in the previous case. Once it reaches the boundary, it waits for the change of the sign of the total force. Assume this happens at the time τ_1. Starting from this time τ_1, we consider one period of the particle motion. Within this period, the particle first moves to the left until it reaches the turning point. This happens at the time $\tau_{min}=\pi-\tau_1$. Thereafter, the force changes its sign again and the particle moves to the right.

We denote by λ^* the force for which the particle just arrives at the left boundary. In this case the amplitude of the mean position of the particle α_{stat} is 0.5 for the first time. This parameter λ^* can be calculated starting from the deterministic equation of motion for an overdamped particle.

Assuming the initial condition $\xi(\tau_1)=1$, with $\tau_1=\arcsin(1/\lambda)$, the solution of the differential equation (14) reads

$$\xi(\tau) = 1 + (\tau-\tau_1) + \lambda\cos(\tau) - \sqrt{\lambda^2-1}. \tag{15}$$

The above-introduced parameter λ^* is determined from Eq. (15) for $\xi(\tau_{min})=0$ resulting in the transcendental equation

$$1+\pi-2\arcsin\left(\frac{1}{\lambda^*}\right)-2\sqrt{(\lambda^*)^2-1}=0. \tag{16}$$

The result is $\lambda^*\approx 1.78$. For an arbitrary value of λ satisfying $1<\lambda\leqslant\lambda^*$, at time τ_2 the particle is at the right boundary again and waits there for the change of the sign of the force at $\tau_3=2\pi+\tau_1$. The time τ_2 is determined by the condition $\xi(\tau_2)=1$ leading to the equation $(\tau_2-\tau_1)+\lambda\cos(\tau_2)-\lambda\cos(\tau_1)=0$. Now the time-averaged mean position is given by

$$\mu_{av} = \frac{1}{\tau_3-\tau_1}\left[\int_{\tau_1}^{\tau_2}\xi(\tau)d\tau + (\tau_3-\tau_2)\right] = \frac{1}{2\pi}\left[2\pi - 1 + \frac{1}{2}(\tau_2-\tau_1)^2 - \sqrt{\lambda^2-1}(\tau_2-\tau_1) + \lambda\sin(\tau_2)\right]. \tag{17}$$

Several values for μ_{av} are listed in Table I.

Case $\lambda>\lambda^*$. As before the particle starts with its motion to the left at time $\tau=\tau_1$ with $1-\lambda\sin(\tau_1)=0$. Now the strength of the oscillating force is so large that the particle impinges on the left boundary at time $\tau=\tau_1'$. It stays on it up to $\tau=\tau_2'$ where the force acting on the particle becomes positive again. At time $\tau=\tau_2$ the particle impinges on the right boundary and waits there for the change of the sign of the force at $\tau=\tau_3=\tau_1+2\pi$.

The solution of Eq. (14) for the motion from the right to the left boundary with the initial condition $\xi_1(\tau_1)=1$ is given by

$$\xi_1(\tau) = 1 + (\tau-\tau_1) + \lambda\cos(\tau) - \sqrt{\lambda^2-1}, \quad \tau_1<\tau<\tau_1'. \tag{18}$$

Similarly for the motion from the left to the right boundary the solution $\xi_2(\tau)$ with the condition $\xi_2(\tau_2')=0$ and $\tau_2'=\pi-\tau_1$ is

$$\xi_2(\tau) = (\tau+\tau_1-\pi) + \lambda\cos(\tau) + \sqrt{\lambda^2-1}, \quad \tau_2'<\tau<\tau_2. \tag{19}$$

Using these equations both times τ_1' and τ_2 can be calculated from

$$\xi_1(\tau_1')=0 \rightsquigarrow 1+(\tau_1'-\tau_1)+\lambda\cos(\tau_1')-\sqrt{\lambda^2-1}=0, \tag{20}$$

$$\xi_2(\tau_2)=1 \rightsquigarrow (\tau_2+\tau_1-\pi)+\lambda\cos(\tau_2)+\sqrt{\lambda^2-1}-1=0. \tag{21}$$

The time-averaged mean position μ_{av} is then determined by

$$\mu_{av} = \frac{1}{\tau_3-\tau_1}\left[\int_{\tau_1}^{\tau_1'}\xi_1(\tau)d\tau + \int_{\pi-\tau_1}^{\tau_2}\xi_2(\tau)d\tau + (\tau_3-\tau_2)\right] = \frac{1}{2\pi}[2\pi-2+\tau_1'-\tau_2+0.5(\tau_1'-\tau_1)^2+\lambda\sin(\tau_1') + \sqrt{\lambda^2-1}(2\tau_1-\tau_1'+\tau_2-\pi)+0.5(\tau_2+\tau_1-\pi)^2 + \lambda\sin(\tau_2)]. \tag{22}$$

Some values for μ_{av} are given in the following Table I.

TABLE II. Amplitude α_{stat} for $\tilde{D}=0$ and for several values of the parameter λ.

λ	1.2	1.5	1.6
α_{stat}	0.08	0.28	0.35

There is a lower limit for the time-averaged mean position of $\lim_{\tilde{D}\to\infty}\mu_{\mathrm{av}}=0.5$. The extrapolation of the numerical μ_{av} values for $\tilde{D}\to 0$, see Fig. 4, is consistent with the μ_{av} values calculated in this subsection and given in Table I.

B. Amplitude of the mean position

Figure 5 shows the amplitude of the mean position α_{stat} as a function of the diffusion coefficient $\tilde{D}$ for various strengths of the oscillating force λ. First we determine α_{stat} for $\tilde{D}=0$ from the deterministic equation of motion (14) analogously to the calculations for the time-averaged mean position:

Case $\lambda \leqslant 1$. The total force acting on the particle is positive or zero all the time. After impinging on the right boundary the particle sticks on it. The amplitude of the mean position in the stationary time limit is zero.

Case $1<\lambda<\lambda^*$. The particle moves to the right. After some time it impinges on the right boundary and remains there up to the time τ_1 when the force becomes negative. Thereafter the particle moves to the left. At time $\tau_{\mathrm{min}}=\pi-\tau_1$ the force changes its sign again and the particle moves to the right.

Starting from the deterministic equation of motion (14) the position of the particle $\xi(\tau)$ is given by

$$\xi(\tau) = 1 + (\tau-\tau_1) + \lambda\cos(\tau) - \lambda\cos(\tau_1), \quad \text{with } \xi(\tau_1)=1, \tag{23}$$

from which the amplitude $\alpha_{\mathrm{stat}}(\tilde{D}=0)$ is determined by

$$\alpha_{\mathrm{stat}}(\tilde{D}=0) = \frac{1}{2}|\xi(\tau_{\mathrm{min}}) - \xi(\tau_1)| = \left|\frac{1}{2}\pi - \tau_1 - \lambda\cos(\tau_1)\right|. \tag{24}$$

The values of the amplitude for several parameters λ are given in Table II.

Case $\lambda \geqslant \lambda^*$. In this case the oscillating force is strong enough to take the particle to the left boundary. The amplitude of the mean position is 0.5. In the rest of Sec. IV B we approximately investigate the amplitude α_{stat} for $\tilde{D}\approx 0$ for various regimes of λ.

Case $\lambda \gg \lambda^*$. Without diffusion the particle moves from the right boundary to the left and stays at the right and left boundaries for certain times τ_r and τ_l, respectively. We assume that for $\tilde{D}\approx 0$ the trajectory of the particle is hardly changed if we replace the time-dependent force during these time intervals τ_r and τ_l by quasistatic force functions, which are functions of the limiting values F_0+F_1 and F_0-F_1, respectively, i.e., $F_r=F_r(F_0+F_1)$ and $F_l=F_l(F_0-F_1)$, and in scaled variables $\lambda^r=\lambda^r(1+\lambda)$ and $\lambda^l=\lambda^l(1-\lambda)$. Now, during the time intervals τ_r and τ_l it is supposed that the mean positions of the particle assume the values of the equilibrium mean positions $\mu_{\mathrm{eq}}=\mu^r_{\mathrm{eq}}$ and $\mu_{\mathrm{eq}}=\mu^l_{\mathrm{eq}}$, respectively,

$$\mu^r_{\mathrm{eq}} = \frac{\tilde{D}}{\lambda^r}\frac{1}{\exp(\lambda^r/\tilde{D})-1}\left[\left(\frac{\lambda^r}{\tilde{D}}-1\right)\exp\left(\frac{\lambda^r}{\tilde{D}}\right)+1\right] \quad \text{and}$$

$$\mu^l_{\mathrm{eq}} = \frac{\tilde{D}}{\lambda^l}\frac{1}{\exp(\lambda^l/\tilde{D})-1}\left[\left(\frac{\lambda^l}{\tilde{D}}-1\right)\exp\left(\frac{\lambda^l}{\tilde{D}}\right)+1\right]. \tag{25}$$

Expressions (25) have been derived by adapting Eq. (9) to the two quasistatic force functions λ^r and λ^l, respectively, and using $\theta=(\omega\Gamma/F_0)l=1$.

Taking into account that $\lambda^r=|\lambda^r|$ and $\lambda^l=-|\lambda^l|$, these equilibrium mean positions can be approximated by

$$\mu^r_{\mathrm{eq}}(\tilde{D}) \approx 1 - \frac{\tilde{D}}{\lambda^r} \quad \text{and } \mu^l_{\mathrm{eq}}(\tilde{D}) \approx -\frac{\tilde{D}}{\lambda^l} = \frac{\tilde{D}}{|\lambda^l|}. \tag{26}$$

Therefore the amplitude of the mean position $\alpha_{\mathrm{stat}}(\tilde{D})$ can be estimated by half of the difference between them,

$$\alpha_{\mathrm{stat}}(\tilde{D}) \approx \frac{1}{2}[\mu^r_{\mathrm{eq}}(\tilde{D}) - \mu^l_{\mathrm{eq}}(\tilde{D})] \approx \frac{1}{2}\left[1 - \tilde{D}\left(\frac{1}{|\lambda^r|}+\frac{1}{|\lambda^l|}\right)\right]. \tag{27}$$

$\alpha_{\mathrm{stat}}(\tilde{D})$ decreases linearly with $\tilde{D}$ with a slope that decreases with increasing strength of the oscillating force. These findings are consistent with the numerical results in Fig. 5 for small values of $\tilde{D}$ and large values of λ, $\lambda \gg \lambda^*$, and represent an extrapolation of the numerical results for $\tilde{D}=0$.

Case $\lambda \approx \lambda^*$. The particle just impinges on the left boundary for $\tilde{D}=0$. Taking into account the diffusion, two corrections of the amplitude have to be introduced, which are caused by the existence of the boundaries. First, the standard deviation of the stochastic process describing the diffusion is proportional to $\sqrt{\tilde{D}}$. Therefore at the left boundary we expect a shift of the mean position by a function $f=f(\sqrt{\tilde{D}})$ caused by cutting off the negative contribution of the noise. Second, with respect to the right boundary we perform analogous considerations as before. In this case $\alpha_{\mathrm{stat}}(\tilde{D})$ is approximately given by

$$\alpha_{\mathrm{stat}}(\tilde{D}) \approx \frac{1}{2}[\mu^r_{\mathrm{eq}}(\tilde{D}) - f(\sqrt{\tilde{D}})] \approx \frac{1}{2}\left[1 - \frac{\tilde{D}}{|\lambda^r|} - f(\sqrt{\tilde{D}})\right]. \tag{28}$$

Both corrections to the amplitude of the mean position are negative. The comparison with $\alpha_{\mathrm{stat}}(\tilde{D})$ for $\lambda=1.8$ in Fig. 5 shows that the amplitude decreases like a root function. Therefore we assume that in this parameter regime of λ the influence of the left boundary is mainly responsible for the decrease of the amplitude with the diffusion coefficient.

Case $1<\lambda<\lambda^*$. To explain the resonance of $\alpha_{\text{stat}}(\tilde{D})$ as a function of $\tilde{D}$ in this parameter range, we estimate the variation of the amplitude as a function of the diffusion coefficent $\Delta\alpha_{\text{stat}}(\tilde{D})=\alpha_{\text{stat}}(\tilde{D}+\epsilon)-\alpha_{\text{stat}}(\tilde{D})$ for $\tilde{D}=0+\delta$, $\delta\ll 1$ and $\epsilon\ll 1$. $\alpha_{\text{stat}}(\tilde{D})$ is approximately given by

$$\alpha_{\text{stat}}(\tilde{D})\approx\frac{1}{2}[\mu_{\text{eq}}^{r}(\tilde{D})-\mu(\tau_{\min})]. \tag{29}$$

$\mu(\tau_{\min})$ is the mean position at the turning point, $\mu_{\text{eq}}^{r}(\tilde{D})$ is the estimated mean position for the rest time τ_r near the right boundary. First we estimate the shift of the mean position at the turning point of the particle. In general the mean position can be expressed by a time-independent part given by the equilibrium mean position for the static force and a time-dependent part containing the perturbation by the external oscillating force, i.e., $\mu(\tau)=\mu_{\text{eq}}+\tilde{\mu}(\tau)$. Now, if the diffusion coefficient increases by ϵ, the equilibrium mean position decreases. We assume that the mean position at the turning point $\mu(\tau_{\min})$ is only shifted by the same amount as the equilibrium mean position μ_{eq} and that there is no additional shift caused by the influence of the left boundary. This assumption is justified by the fact that for $\tilde{D}=0+\delta$ the width of the probability density of the particle at the turning point is arbitrarily small and given by the parameter δ. The left boundary causes a shift of the mean position only when the particle contacts the left boundary. To estimate the variation of the amplitude $\Delta\alpha_{\text{stat}}(\tilde{D})$ we consider the dependence of the function $\mu_{\text{eq}}^{\Delta}(\tilde{D})=\mu_{\text{eq}}^{r}(\tilde{D})-\mu_{\text{eq}}(\tilde{D})$ on the diffusion coefficient $\tilde{D}$, with $\mu_{\text{eq}}^{r}(\tilde{D})$ and $\mu_{\text{eq}}(\tilde{D})$ given by

$$\mu_{\text{eq}}^{r}(\tilde{D})=\frac{\tilde{D}}{\lambda^{r}}\frac{1}{\exp(\lambda^{r}/\tilde{D})-1}\left[\left(\frac{\lambda^{r}}{\tilde{D}}-1\right)\exp\left(\frac{\lambda^{r}}{\tilde{D}}\right)+1\right]\quad\text{and}$$

$$\mu_{\text{eq}}(\tilde{D})=\tilde{D}\frac{1}{\exp(1/\tilde{D})-1}\left[\left(\frac{1}{\tilde{D}}-1\right)\exp\left(\frac{1}{\tilde{D}}\right)+1\right]. \tag{30}$$

As seen in Fig. 7, $\mu_{\text{eq}}^{\Delta}(\tilde{D})$ is an increasing function for $\tilde{D}=0+\delta$, the larger λ the larger the slope of this function. Thus the amplitude of the mean position is an increasing function for $\tilde{D}=0+\delta$ and $\lambda<\lambda^*$. The width of the probability density at the turning point can always be reduced by decreasing the parameter δ so that the particle does not contact the left boundary. We have thus an increase of α_{stat} for small values of $\tilde{D}$ and, as discussed at the end of Sec. III C, a decrease of α_{stat} for large values of $\tilde{D}$. In between there must be at least one maximum. Therefore there is a resonance for all $\lambda<\lambda^*$ and from this it follows that $\lambda^{\text{crit},\alpha}$ is equal to λ^*.

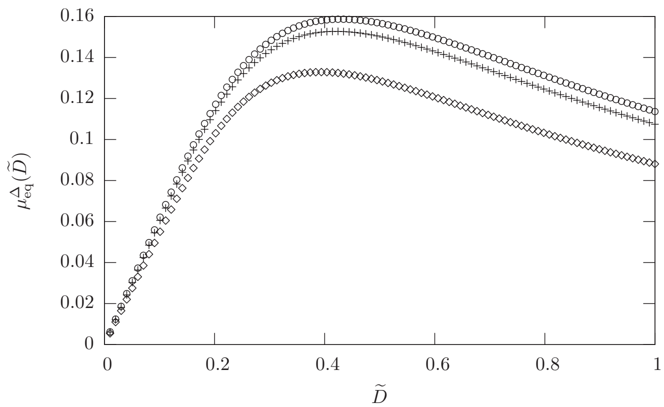

FIG. 7. The quantity $\mu_{\text{eq}}^{\Delta}(\tilde{D})=\mu_{\text{eq}}^{r}(\tilde{D})-\mu_{\text{eq}}(\tilde{D})$ as a function of the parameter $\tilde{D}$ for three values of the parameter $\lambda_r=1+\lambda$: $\lambda=1.6$ (○); $\lambda=1.5$ (+); $\lambda=1.2$ (◇).

Case $\lambda\leq 1$. The amplitude of the mean position is zero for $\tilde{D}=0$ as well in the limit $\tilde{D}\to\infty$, because in this limit the diffusion process dominates the deterministic motion. Therefore there has to be a maximum in this parameter range λ, too.

Summarizing Sec. IV B, we have calculated the stationary amplitude of the mean position for $\tilde{D}=0$ by solving the deterministic equation of motion. Then as seen in Fig. 5, we have shown that the calculated values of $\alpha_{\text{stat}}(\tilde{D}=0)$ given in Table II are comparable to those values, which we would expect after extrapolating the numerical values for $\tilde{D}\to 0$. Furthermore, our considerations show that there has to be a resonance of the amplitude α_{stat} for all parameters $\lambda<\lambda^*$. For $\lambda=\lambda^*$ the particle impinges on the left boundary for the first time. Increasing the parameter λ the amplitude of the mean position first decreases like a root function, if the parameter λ is more increased the amplitude of the mean position decreases linearly with $\tilde{D}$.

C. Dynamical shift

Figure 6 presents the dependence of the dynamical shift $\sigma=\mu_{\text{av}}-\mu_{\text{eq}}$ on the diffusion coefficient $\tilde{D}$. For $\tilde{D}=0$ the dynamical shift can be calculated from the values of the time-averaged mean position μ_{av} given in Sec. IV A and the equilibrium position $\mu_{\text{eq}}=1$. These values are listed in Table III. To discuss the dependence of the dynamical shift on the diffusion coefficient we make qualitative estimates analogously to the considerations for the amplitude of the mean position.

Case $\lambda\leq 1$. The dynamical shift is zero for $\tilde{D}=0$ as well as in the limit $\tilde{D}\to\infty$. In both cases the values of μ_{av} and μ_{eq}

TABLE III. Dynamical shift σ for $\tilde{D}=0$ and for several values of the parameter λ.

λ	0.5	1.0	1.2	1.5	1.6	1.8	2.5	3.0	5.0
σ	0.00	0.00	−0.03	−0.13	−0.18	−0.28	−0.36	−0.38	−0.43

are the same. Therefore there has to be a minimum in this parameter range λ.

Case $1<\lambda<\lambda^*$. We approximate the variation of the time-averaged mean position $\Delta\mu_{\mathrm{av}}(\tilde{D})$ [see text before Eq. (29)],

$$\Delta\mu_{\mathrm{av}}(\tilde{D}) = \mu_{\mathrm{av}}(\tilde{D}+\epsilon) - \mu_{\mathrm{av}}(\tilde{D}), \tag{31}$$

for $\tilde{D}=0+\delta$, $\delta \ll 1$, and $\epsilon \ll 1$. Multiplying the FPE (11) by the space variable ξ, integrating over the domain, and taking into account the boundary conditions we get the following equation for the mean position:

$$\frac{\partial}{\partial\tau}\mu(\tau) = -\tilde{D}[u(1,\tau)-u(0,\tau)] + f(\tau) \tag{32}$$

with $f(\tau)=1-\lambda\sin(\tau)$. $u(1,\tau)$ and $u(0,\tau)$ are the probability densities at the boundaries. We expand the time derivative of the variation of the mean position in powers of ϵ about $\tilde{D}$,

$$\frac{\partial}{\partial\tau}\Delta\mu^{\tilde{D}}(\tau) = \frac{\partial}{\partial\tau}[\mu^{\tilde{D}+\epsilon}(\tau) - \mu^{\tilde{D}}(\tau)] = -(\tilde{D}+\epsilon)u^{\tilde{D}+\epsilon}(1,\tau) + \tilde{D}u^{\tilde{D}}(1,\tau) + (\tilde{D}+\epsilon)u^{\tilde{D}+\epsilon}(0,\tau) - \tilde{D}u^{\tilde{D}}(0,\tau) \tag{33}$$

and get the following equation:

$$\frac{\partial}{\partial\tau}\Delta\mu^{\tilde{D}}(\tau) = -\epsilon\left[u^{\tilde{D}}(1,\tau) + \tilde{D}\frac{\partial}{\partial\tilde{D}}u^{\tilde{D}}(1,\tau)\right] + \epsilon\left[u^{\tilde{D}}(0,\tau) + \tilde{D}\frac{\partial}{\partial\tilde{D}}u^{\tilde{D}}(0,\tau)\right]. \tag{34}$$

Starting from the time τ_1 (for the definitions of the various times, see Sec. IV A), we consider again one period $\tilde{\tau}$ of the motion of the particle. At time τ_1 the total force becomes negative and the particle moves to the left. Now we calculate the variation of the time averaged mean position using partial integration in the transition to the last equality of the following equation:

$$\begin{aligned}
\Delta\mu_{\mathrm{av}}(\tilde{D}) &= \frac{1}{\tilde{\tau}}\int_{\tau_1}^{\tau_1+\tilde{\tau}} d\tau\,\Delta\mu^{\tilde{D}}(\tau) \\
&= \frac{1}{\tilde{\tau}}\int_{\tau_1}^{\tau_1+\tilde{\tau}} d\tau\left[\Delta\mu^{\tilde{D}}(\tau_1) + \int_{\tau_1}^{\tau} d\tau'\frac{\partial}{\partial\tau'}\Delta\mu^{\tilde{D}}(\tau')\right] \\
&= \Delta\mu^{\tilde{D}}(\tau_1) - \epsilon\frac{1}{\tilde{\tau}}\int_{\tau_1}^{\tau_1+\tilde{\tau}} d\tau\int_{\tau_1}^{\tau} d\tau'\left[u^{\tilde{D}}(1,\tau') + \tilde{D}\frac{\partial}{\partial\tilde{D}}u^{\tilde{D}}(1,\tau')\right] + \epsilon\frac{1}{\tilde{\tau}}\int_{\tau_1}^{\tau_1+\tilde{\tau}} d\tau\int_{\tau_1}^{\tau} d\tau'\left[u^{\tilde{D}}(0,\tau') + \tilde{D}\frac{\partial}{\partial\tilde{D}}u^{\tilde{D}}(0,\tau')\right] \\
&= \Delta\mu^{\tilde{D}}(\tau_1) - \epsilon\frac{1}{\tilde{\tau}}\int_{\tau_1}^{\tau_1+\tilde{\tau}} d\tau(\tau_1+\tilde{\tau}-\tau)\left[u^{\tilde{D}}(1,\tau) + \tilde{D}\frac{\partial}{\partial\tilde{D}}u^{\tilde{D}}(1,\tau)\right] + \epsilon\frac{1}{\tilde{\tau}}\int_{\tau_1}^{\tau_1+\tilde{\tau}} d\tau(\tau_1+\tilde{\tau}-\tau)\left[u^{\tilde{D}}(0,\tau) + \tilde{D}\frac{\partial}{\partial\tilde{D}}u^{\tilde{D}}(0,\tau)\right].
\end{aligned} \tag{35}$$

Now we approximate the three contributions to $\Delta\mu_{\mathrm{av}}(\tilde{D})=h_1(\tilde{D})+h_2(\tilde{D})+h_3(\tilde{D})$ as follows: The first term $h_1(\tilde{D})$ is approximated by the difference of the equilibrium mean positions for the quasistatic force λ^r [see Eq. (26)],

$$\Delta\mu^{\tilde{D}}(\tau_1) = \mu_{\mathrm{eq}}^{r,\tilde{D}+\epsilon} - \mu_{\mathrm{eq}}^{r,\tilde{D}} \approx -\frac{\epsilon}{\lambda^r}. \tag{36}$$

To obtain the sign of the second part $h_2(\tilde{D})$ we assume that for the time interval $[\tau_1,\tau_2]$ the probability density at the right boundary is approximately zero. For the time interval $[\tau_2,\tilde{\tau}]$ we take for the probability density at the right boundary $u^{\tilde{D}}(1,\tau)$ the equilibrium probability density for the quasistatic force, i.e., we replace in Eq. (11) the term $[1-\lambda\sin(\tau)]$ by λ^r and use the same procedure as for the unscaled equation (9),

$$u^{\tilde{D}}(1,\tau) \approx \frac{\lambda^r}{\tilde{D}}\frac{1}{[\exp(\lambda^r/\tilde{D})-1]}\exp\left(\frac{\lambda^r}{\tilde{D}}\right). \tag{37}$$

Thus we get

$$\left[u^{\tilde{D}}(1,\tau) + \tilde{D}\frac{\partial}{\partial\tilde{D}}u^{\tilde{D}}(1,\tau)\right] \approx \left(\frac{\lambda^r}{\tilde{D}}\right)^2\exp\left(-\frac{\lambda^r}{\tilde{D}}\right)\frac{1}{[1-\exp(-\lambda^r/\tilde{D})]^2}. \tag{38}$$

This is a positive function so that the second term $h_2(\tilde{D})$ as well as the first term $h_1(\tilde{D})$ give a negative contribution to the variation of the time-averaged mean position. In the parameter range $\tilde{D}=0+\delta$ we can always make the width of the probability density arbitrarily small so that the particle does not contact the left boundary for all parameters $\lambda<\lambda^*$. With this assumption we can neglect the probability density at the left boundary so that the third term $h_3(\tilde{D})$ can be assumed as zero. To get the variation of the dynamical shift we have to calculate the variation of the equilibrium mean position $\Delta\mu_{\rm eq}^{\tilde{D}}$, too. This variation is approximately given by [see Eq. (12)]

$$\Delta\mu_{\rm eq}^{\tilde{D}} = \mu_{\rm eq}^{\tilde{D}+\epsilon} - \mu_{\rm eq}^{\tilde{D}} \approx -\epsilon. \tag{39}$$

For $\lambda < \lambda^*$ the variation of the time-averaged mean position gives a negative contribution to the variation of the dynamical shift whereas the contribution of the variation of the equilibrium mean position is positive. On the basis of our numerical results we can conclude that the variation of the time-averaged mean position has to be larger than the variation of the equilibrium mean position to make the dynamical shift a decreasing function for $\tilde{D}=0+\delta$. The dynamical shift has to assume a minimum in this parameter range λ. If $\lambda \geqslant \lambda^*$ the particle impinges on the left boundary in the deterministic motion. For this reason the probability density at the left boundary will be not negligible if the diffusion process is switched on. So we expect an additional positive contribution from $h_3(\tilde{D})$ to the variation of the dynamical shift and we assume that the minimum disappears. Thus the parameter $\lambda^{\mathrm{crit},\sigma}$ determining the critical force is equal to λ^* and identical to $\lambda^{\mathrm{crit},\alpha}$.

Case $\lambda \geqslant \lambda^*$. For $\tilde{D}=0+\delta$ the mean positions of the particle take approximately the equilibrium values $\mu_{\rm eq}^{r,\tilde{D}} \approx 1-\frac{\tilde{D}}{\lambda^r}$ and $\mu_{\rm eq}^{l,\tilde{D}} \approx -\frac{\tilde{D}}{\lambda^l}$ [see Eq. (26)] during the times τ_r and τ_l close to the right and the left boundaries. The variation of the dynamical shift is approximately given by

$$\begin{aligned}\Delta\sigma(\tilde{D}) &\approx \frac{1}{\tilde{\tau}}\left[\int_0^{\tau_r} d\tau(\Delta\mu^{\tilde{D}}(\tau) - \Delta\mu_{\rm eq}^{\tilde{D}}) + \int_0^{\tau_l} d\tau(\Delta\mu^{\tilde{D}}(\tau) - \Delta\mu_{\rm eq}^{\tilde{D}})\right] \\ &\approx \frac{1}{\tilde{\tau}}\left[\int_0^{\tau_r} d\tau(\Delta\mu_{\rm eq}^{r,\tilde{D}} - \Delta\mu_{\rm eq}^{\tilde{D}}) + \int_0^{\tau_l} d\tau(\Delta\mu_{\rm eq}^{l,\tilde{D}} - \Delta\mu_{\rm eq}^{\tilde{D}})\right].\end{aligned} \tag{40}$$

Therefore with

$$\Delta\mu_{\rm eq}^{r,\tilde{D}} = \mu_{\rm eq}^{r,\tilde{D}+\epsilon} - \mu_{\rm eq}^{r,\tilde{D}} \approx -\frac{\epsilon}{\lambda^r} \quad \text{and}$$

$$\Delta\mu_{\rm eq}^{l,\tilde{D}} = \mu_{\rm eq}^{l,\tilde{D}+\epsilon} - \mu_{\rm eq}^{l,\tilde{D}} \approx -\frac{\epsilon}{\lambda^l} \tag{41}$$

and $\Delta\mu_{\rm eq}^{\tilde{D}} = \mu_{\rm eq}^{\tilde{D}+\epsilon} - \mu_{\rm eq}^{\tilde{D}} \approx -\epsilon$ [see Eq. (12)] the variation of the dynamical shift is given by

$$\Delta\sigma(\tilde{D}) \approx \epsilon\frac{\tau_r}{\tilde{\tau}}\left[1-\frac{1}{\lambda^r}\right] + \epsilon\frac{\tau_l}{\tilde{\tau}}\left[1-\frac{1}{\lambda^l}\right]. \tag{42}$$

The variation of the dynamical shift is always positive. Therefore for these λ values the dynamical shift is an increasing function and the minimum is lost.

The dynamical shift is always negative. Our estimates show that for all $\lambda < \lambda^*$ the dynamical shift as a function of $\tilde{D}$ has a minimum. The parameter value λ^* corresponds to the force strength for which the particle impinges on the left boundary for the first time. If the parameter λ is larger than λ^* the dynamical shift always increases with increasing diffusion coefficient.

V. CONCLUSION

We have considered the diffusion of an overdamped particle in a one-dimensional space domain with two reflecting boundaries. A static force pushing the particle to the right as well as a harmonically oscillating force are acting on it. In the numerical part of this paper we have presented the mean position of the particle for different diffusion coefficients and different strengths of the external oscillating force as a function of time. In the stationary regime the mean position is a periodic function with the period of the external force. Obviously by varying the diffusion coefficient (i.e., the temperature) and the external force strength the motion of the particle in the domain can be restricted. We have discussed the dependence of the amplitude of the mean position $\alpha_{\rm stat}$ on the diffusion coefficient $\tilde{D}$ in detail. There is a resonance of the amplitude for a certain parameter range of the external force strength. If the strength of the external force increases beyond a critical value, the resonance is lost. Our qualitative estimates show that the critical strength of the force is given by that external force strength for which the particle impinges on the left boundary for the first time in the deterministic motion. For higher force strengths than the critical one the amplitude of the mean position decreases with increasing diffusion coefficient. Finally, we have also considered the dynamical shift σ of the mean position of the particle. This quantity describes the following feature of the response. In the stationary regime, the time-averaged mean position of the particle is shifted against the slope of the unperturbed potential as compared to the equilibrium mean position in the problem with the static force alone. Differently speaking, in the stationary nonequilibrium regime, the external symmetrically oscillating force produces a permanent excess of the time-averaged system internal energy, as compared to the equilibrium internal energy in the problem with the static force alone. Our numerical results display for the dynamical shift a similar resonancelike behavior as for the amplitude of the mean position. There is also a critical force strength for which the resonance is lost. In the case of smaller external force strengths than the critical one there is a diffusion coefficient (or temperature) for which a maximum of energy is trapped in the system. We find that the critical force strength of the dynamical shift has to be the same as for the amplitude of the mean position. In both cases the resonance disappears as soon as the second boundary influences the motion of the particle. Therefore we expect a resonance related to the diffusion coefficient in the amplitude as well as in the dynamical shift independent on the strength of the external oscillating force if the particle moves in a domain with only one reflecting boundary. We hope that our discussion will stimulate experimental investigations along the lines as mentioned in the Introduction.

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